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COMMON FIXED POINT THEOREMS FOR HYBRID PAIRS OF COMPATIBLE TYPE (N) MAPS

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ABSTRACT

We obtain common fixed point theorems for compatible type (N) maps of hybrid pair which is more general condition for existence of common fixed point theorems. We improve the results on occasionally weakly compatible maps of hybrid pairs. 2000 Mathematics subject classification: 47H10; 54H25.

KEYWORDS: Weakly commuting maps, compatible maps of type(N),weakly compatible maps, common fixed point theorem

Let (X, d) denotes a metric space and CB(X) the family of all nonempty closed and bounded subsets of X. Let H be the Hausdorff metric on CB(X) induced by the metric d; i.e.,

$H(A, B) = \max \{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\}$

For A, B in CB(X), where $d(x, A) = \inf \{d(x, y): y \in A\}$.

Let f, g be two self-maps of a metric space (X, d). In his paper, Sessa (1982) defined f and g to be weakly commuting if for all $x \in X$

$d(fgx, gfx) \leq d(gx, fx)$.

It can be seen that two commuting maps ($fgx = gfx, \forall x \in X$) are weakly commuting, but the converse is false in general. (Sessa , 1982) .Afterwards Jungck in 1986 extended the concepts of commutativity and weak commutativity by giving notion of compatibility. Maps f and g above are compatible if

$\lim_{n\to\infty} d(fgx_n,gfx_n)=0$

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} f x_n = \lim_{n\to\infty} g x_n = t$ for some $t \in X$.

Obviously, weakly commuting maps are compatible, but the converse is not true in general. (Jungck , 1986)

Further, Kaneko H. and Sessa in 1989 extended the concept of compatibility for single valued maps to the setting of single and multi-valued maps as follows: $f : X \to X$ and $F : X \to CB(X)$ are said to be compatible if $fFx \in CB(X)$ for all $x \in X$ and $\lim_{n\to\infty} H(Ffx_n, fFx_n) = 0$, Whenever $\{x_n\}$ is a sequence in X such that $Fx_n \to A \in CB(X)$ and $fx_n \to t \in A$.

Definition 1.1. Let (X,d) be a metric space and $f : X \to X$ and $F : X \to CB(X)$. Then a point $x \in X$ is called a coincidence point of f and F if $fx \in Fx$. We shall call $w = fx \in Fx$ a point of coincidence of f and F.

Jungck G. and Rhoades (1998) weakened the notion of compatibility for single and multi-valued maps by giving the concept of weak compatibility. They define maps f and F above to be weakly compatible if they commute at their coincidence points; i.e., if $fFx = Ffx$ whenever $fx \in Fx$.

Shrivastava et al. (2000) gave another generalization of compatibility for single and multi-valued maps by giving the concept of compatibility of type (N). They define pair (f,F) [where f: $X \to X$, F: $X \to CB(X)$] is said compatible type (N) if $fx \in Fx \Rightarrow ffx \in Ffx$.

Abbas and Rhoades (2007) generalized the concept of weak compatibility in the setting of single and multi-valued maps by introducing the notion of occasionally weak compatibility (owc). Maps f and F are said to be owc if and only if there exists some point x in X such that

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$fx \in Fx$ and $fFx \subseteq Ffx$.

Rao et al. (2007) The hybrid pair (f,F) is said to be Occasionally Weakly Semi compatible (o w s c) if there exits some point $x \in X$ such that $fx \in Fx$ and $ffx \in Ffx$.

Definition 1.2. Let X be a metric space and $f: X \to X$ and $F: X \to CB(X)$. Then a point $x \in X$ is called a common fixed point of f & F if $x = fx \in Fx$.

Our theorems are proved in symmetric spaces which are more general than metric spaces.

Definition1.3. Let X be a set. A symmetric on X is a mapping $r: X\times X \to [0,\infty]$ such that $r(x,y) = 0$ if $x = y$ & $r(x,y) = r(y,x)$ $\forall x,y \in X$. Let B(X) be the class of all non-empty bounded subsets of X, denote $\delta(A, B) = \sup\{r(a,b) : a \in A, b \in B\}$ and $D(A,B) = \inf\{r(a,b): a \in A\ b \in B\}$.

Example1.4. Let $X = \{0,1\}$ and d be an usual metric. Define $f0=1$, $f1= 0$ and $Fx = \{0, 1\}$. Then (f, F) is owc hybrid pair. It is clear that f and F have no common fixed point.

Remark 1.5. We observed that all results in Abbas and Rhoades (2007) are not valid in view of Example 1.4 However they are valid if Hausdroff metric H is replaced by δ

RESULTS

Concept of compatibility of type (N) is more general form weak compatibility and occasionally weak compatibility.

Lemma 2.1. Concept of Weak compatibility (wc) and occasionally Weak compatibility (owc) for hybrid pair implies Compatibility of type (N) but not conversely.

Proof. Let (f,F) is occasionally Weakly compatibile pair. Pair (f,F) is o w c if there exists some point x in X such that $fx \in Fx$ and $fFx \subseteq Ffx$.

$$\Rightarrow ffx \in Ffx$$

Thus $fx \in Fx \Rightarrow ffx \in Ffx$

i.e. (f,F) is compatible of type (N).

Similarly if (f, F) is weak compatible pair.

Then $fFx = Ffx$, whenever $fx \in Fx \Rightarrow ffx \in Ffx$

i.e. (f,F) is compatible of type (N).

But this example shows converse is not true.

Example 2.2. Let $X = [0,1]$ and d be an usual metric.

Define $fx = 1-x$ and $Fx = [0, ½]$ Then $f(½) = ½ \in F(½)$ But $fF(½) = [½,1] \not\subset Ff(½) = [0, ½]$ And $ff(½) = ½ \in Ff(½)$

Thus pair (f,F) is neither w c nor owc but compatible type (N).

Now we extend lemma 1 of Jungck and Rhoades (2006) for hybrid pair.

Lemma 2.3. Let X be a set and $f : X \to X$ and $F : X \to CB(X)$. Let (f,F) be pair of compatible type(N) maps. If f and F have a unique point of coincidence $w = fx \in Fx$ then w is unique common fixed point of f and F.

Proof. Let $x \in X$ is coincidence point and w is unique point of coincidence of f and F then $w = fx \in Fx$. Since (f, F) is of compatible type (N) therefore $w = fx \in Fx \Rightarrow ffx \in Ffx$ which says that ffx is also a point of coincidence of f & F.

Since the point of coincidence $w = fx$ of f & F is unique by hypothesis therefore, $w = fx = ffx \in Ffx$

Thus $w = fx$ is a common fixed point of f & F. Moreover, if z is any common fixed point of f & F then, $w = z = fz \in Fz$ by the uniqueness of the point of coincidence, i.e. w is a unique point of coincidence f & F then w is unique common fixed point f and F.

Remark 1. In view of lemma 1 and example 1 all theorems and corollaries of Bouhadjera et al. (2008) can generalized taking compatible pair of type (N) in the place of owc maps.

Remark 2. In view of lemma 1 and example 1 all theorems and corollaries of Abbas and Rhoades (2007) can generalized taking compatible pair of type(N) in the place of o w c maps. Also due to remark 1.4 of Rao (2007) Hausdorff metric H is replaced by δ

We introduce Multi-valued version of Theorem 1 of G. Jungck (1996)

Theorem 2.4. Let X be a set with a symmetric r. Suppose that f, g are Self maps of X and $S, T: X \to B(X)$ and the pairs (f, S) and (g, T) are compatible of type (N) and there exists points x and y in X such that $fx \in Fx$ and $gy \in Gy$. If

$$\delta(Sx, Ty) < M(x, y) \quad \ldots\ldots\ldots\ldots\ldots (1)$$

for each $x, y \in X$ for which $fx \neq gy$.

$M(x, y) = \max\{r(fx, gy), D(Sx, fx), D(Ty, gy), D(Sx, gy), D(Ty, fx)\}$

Then there is a unique point $w \in X$ such that $w = fw \in Sw$ and a unique point $z \in X$, such that $z = gz \in Tz$. Moreover $z = w$, so that there is a unique common fixed point of f, g, S and T.

Proof. Since the pairs (f, S) and (g, T) are each compatible of type (N) and there exist points x, y $\in$X such that fx $\in$ Fx and gy $\in$Ty.

We claim that fx = gy For other wise by (1) we get

r(fx , gy) =≤(Sx,Ty) < M(x , y) and

M(x ,y) = max {r (fx , gy) D(Sx , fx), D(Ty ,gy), D(Sx , gy), D(Ty , fx)} = max {r (fx , gy) , r(fx , fx) ,r(gy ,gy) , r(fx , gy) , r(gy , fx)}= r(fx, gy)

Thus r(fx , gy) < M(x , y) = r(fx , gy)

Which is a contradiction. Therefore fx = gy

i.e. fx $\in$ Sx , fx = gy , gy $\in$ Ty .

Moreover ,if there is another point z such that fz $\in$ Sz then using (1) it follows that fz $\in$Sz , gy $\in$Ty, fz = gy or fx = fz and w = fx $\in$Sx is unique point of coincidence of f and S by Lemma (2.3) w is only common fixed point of f and S.i.e.w = fw $\in$Sw. By symmetry there is unique point z $\in$ X such that z = gz $\in$ Tz

Suppose that w$\in$ z using (1) we get

r (w , z) = r (fw , gz) < M (x ,y) = r (w ,z)

Which is a contradiction. Therefore w = z and w is common fixed point by lemma (2.3) it is clear that w is unique.

Corollary 2.5. Let X be a set with symmetric r suppose that f , g : X $\in$ X and S ,T : X $\in$ B (X) and pairs (f ,S) and (g ,T) are compatible type N.

If δ (Sx , Ty) ≤ hm (x , y) …………….. (2)

Where (x, y) = max {r(fx , gy), D(Sx , fx), D(Ty ,gy), [D(Sx , gy)s+D(Ty, fx)] /2} and 0≤h < 1Then there is a unique point w $\in$ X such that w = fw $\in$ Sw and a unique point z $\in$ X , such that z = gz $\in$Tz . Moreover z = w , so that there is a unique common fixed point of f, g, S and T, then f,g,S,T have unique common fixed point.

Proof. Since (2) is special case of (1) the result follows immediately from theorem (1).

Remark 3. Similarly we can generalized all theorems and corollaries of Abbas and Rhoades (2008) by taking compatible pair of type (N) in the place of o w c maps.

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